Reduced Basis Modeling for Uncertainty Quantification of Electromagnetic Problems in Stochastically Varying Domains

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Summary. The reduced basis method (RBM) is a model order reduction technique for parametrized partial differential equations (PDEs) which enables fast and reliable evaluation of the transfer behavior in many-query and real-time settings. We use the RBM to generate a low order model of an electromagnetic system governed by time-harmonic Maxwell's equations. The reduced order model then makes it feasible to analyze the uncertainty by a Monte Carlo simulation. Stochastic collocation is employed as a second technique to estimate the statistics.

1 Introduction

As the simulation of integrated circuits requires a significant amount of computational power, the simulation of large-scale models benefits greatly from using model order reduction (MOR) techniques. The original system size is typically reduced to a dimension of less than 100, which allows to examine the frequency response of parametrized systems using the reduced model.

Of particular interest are small random variations in geometry, which are given by random variables of given distribution. Inaccuracies in the production process can lead to variations in the geometry, which influence the overall behavior of the model. The influence of the geometric variations on quantities of interest is measured in the form of expectation and variance of output functionals. This is summarized under the term Uncertainty Quantification (UQ).

As a sample application we consider a coplanar waveguide, which is governed by Maxwell's equations. The parametric model reduction technique we use is the reduced basis method (RBM). The RBM generates low order models to parametrized partial differential equations (PDEs) and the approximation tolerance is certified with rigorous error estimators.

2 Reduced Basis Model Order Reduction for Maxwell's Equations

The coplanar waveguide (see [2]) is bounded by a shielded box with perfect electric conducting (PEC) boundary. The system is excited by discrete ports used to model input and output currents/voltages.

We are interested in parameter studies of the input-output behavior of electromagnetic models. Therefore, we need to compute the electromagnetic field induced by the applied current. We simulate Maxwell's equations in the second order time-harmonic formulation

$$\nabla \times \mu^{-1} \nabla \times E + j\omega\sigma E - \omega^2 \varepsilon E = -j\omega J,$$

subject to zero boundary conditions

$$E \times n = 0 \quad \text{on} \quad \Gamma_{\text{PEC}},$$

which is solved for the discretized electric field E. The equation is discretized with Nédélec finite elements (see [1]) over the entire shielded box as the computational domain. The parameter vector is denoted by $\nu \in \mathscr{D} \subset \mathbb{R}^p$, such that $E(\nu)$ is the parameter-dependent electric field solution.

MOR allows to significantly reduce the computational time required for parameter studies. It substitutes the large-scale model by a model of low order, which approximates the transfer behavior. The aim of the RBM is to determine a low order space X_N of dimension N, which approximates the parametric manifold $M^\nu = \{E(\nu) | \nu \in \mathscr{D}\}$ well. Assuming sufficient smoothness of M^ν, a space X_N can be determined, such that projecting the variational form onto X_N gives good approximations $E_N(\nu)$ to $E(\nu)$. The space X_N is spanned by snapshots of the field solutions for a discrete set of parameter realizations. The snapshot locations are chosen in a greedy process using a rigorous error estimator. The error estimators $\Delta_N(\nu)$, which give rigorous bounds on the approximation error in the $H(\text{curl})$ norm:

$$\|E(\nu) - E_N(\nu)\|_{H(\text{curl})} \leq \Delta_N(\nu),$$

are used to certify the accuracy of the reduced order model. See [2] for more details.

As in UQ, we are interested in the expected value, standard deviation or k-σ values of particular quantities of interest. As these quantities of interest are given by functionals $l(E)$ of the field solution, the accuracy of the reduced order model w.r.t. $l(E)$ can be further enhanced by using a primal-dual error estimation framework. Here, the adjoint system equation is solved as well and the output error estimator $\Delta_N^o(\nu)$ is given by

$$\Delta_N^o(\nu) = \frac{\|r^{pr}(\cdot;\nu)\|_{X'}\|r^{du}(\cdot;\nu)\|_{X'}}{\beta_N(\nu)},$$

where r^{pr} is the primal, r^{du} the dual residuum and β_N an estimator for the inf-sup stability constant. The dual space of the full order finite element space is denoted by X', see [2] for more details.

3 Uncertainty Quantification

Let $(\Omega, \mathscr{F}, \mathscr{P})$ denote a probability space. Given is a square integrable random variable $Y : \Omega \to \mathbb{R}$ with probability density function f and a function $g : \Gamma \to \mathbb{R}^d$, corresponding to a mapping of realizations of a random variable to the output of the electromagnetic system.

Stochastic collocation computes statistical quantities like the mean by a quadrature rule

$$\mathbb{E}(g(Y)) = \int_\Gamma g(x) f(x) dx \approx \sum_{i=1}^{n} g(\xi_i) w_i,$$

where the realizations ξ_i are the sample points, n denotes the sample size and the weights w_i are determined using the probability density function f. See [3] for more details.

In statistical analysis the expectation and variance of quantities of interest like the response surface w.r.t. uncertain parameters is computed.

Monte Carlo simulations use equally weighted samples, which have been generated using the underlying distribution. A drawback of the Monte Carlo simulation is its slow convergence rate of $1/\sqrt{n}$. Additionally, the stochastic collocation is employed.

We use stochastic collocation in sparse grids of the Stroud- or Hermite-type. Anisotropic sparse grids can give additional computational advantages over isotropic grids, see [5]. To further enhance the computation speed of statistical quantities, stochastic collocation is combined with reduced basis model order reduction. This allows to quantify models of a much larger complexity.

4 Modeling Stochastically Varying Domains

For all $\omega \in \Omega$, let $D(\omega)$ denote the random domain with boundary $\partial D(\omega)$. We employ a mapping to a deterministic domain $\overline{D}$ such that we can assemble the system matrices for the domain $\overline{D}$ and use affine transformations to map to a particular realization $D(\omega)$. In [6] the affine transformations are shown for a deterministic parameter. For our analysis, we quantize the geometry into subsections and allow stochastic variations on each subsection. The affine transformation is then applied to each subsection.

5 Numerical Experiments

Using the RBM on parametric systems with deterministic parameters shows exponential convergence rates, see Fig. 1. Similarly to [4], we aim to extend this to stochastic parameters using the primal-dual error estimation framework but we focus on the particular application to Maxwell's equations. The talk will cover numerical results when applying the RBM in combination with Monte Carlo simulation and stochastic collocation.

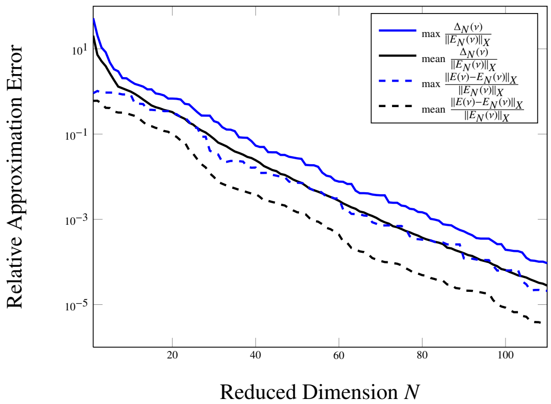

Fig. 1. Convergence of the RBM on the coplanar waveguide with two parameters. The full model contains 52'134 degrees of freedom.

References

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