Multirate GARK schemes for multiphysics problems

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Summary. Multirate GARK schemes define a multirate extension of GARK schemes, generalized additive Runge-Kutta schemes. These allow for exploiting mul- tirate behaviour in both the right-hand sides and in the components in a rather general setting, and are thus especially useful for coupled problems in a multiphysics setting.

We apply MGARK schemes to a benchmark example from thermal-electrical coupling, characteried by a slow and fast part with a stiff and non-stiff characteristic, resp. We test two MGARK schemes: an IMEX method, which makes fully use of the different dynamics and stability properties of the coupled system; and a fully implicit schemes, which inherits the stability properties from both underlying schemes without any coupling constraint.

1 Multirate GARK schemes

We consider a two-way partitioned system

$$y' = f(y) = f^{\{\mathfrak{s}\}}(y) + f^{\{\mathfrak{f}\}}, \quad y(t_0) = y_0\,, \qquad (1)$$

with one slow component $\{\mathfrak{s}\}$, and one active (fast) component $\{\mathfrak{f}\}$. Note that this setting contains component-wise splitting

$$y = \begin{pmatrix} y_\mathfrak{s} \\ y_\mathfrak{f} \end{pmatrix}, \quad f^\mathfrak{s} = \begin{pmatrix} f_\mathfrak{s} \\ 0 \end{pmatrix}, \quad f^\mathfrak{f} = \begin{pmatrix} 0 \\ f_\mathfrak{f} \end{pmatrix}$$

as a special case.

The slow component is solved with a large step H, and the fast one with small steps $h = H/M$. We will consider the multirate generalization of GARK schemes [3] with M micro steps $h = H/M$, as given in the following

Definition 1 (Multirate GARK method [4]). *One macro-step of a generalized additive multirate Runge-Kutta method with M equal micro-steps reads*

$$\begin{aligned}
Y_i^{\{\mathfrak{s}\}} &= y_n + H \sum_{j=1}^{s^{\{\mathfrak{s}\}}} a_{i,j}^{\{\mathfrak{s},\mathfrak{s}\}} f^{\{\mathfrak{s}\}}\left(Y_j^{\{\mathfrak{s}\}}\right) + \\
&\quad + h \sum_{\lambda=1}^{M} \sum_{j=1}^{s^{\{\mathfrak{f}\}}} a_{i,j}^{\{\mathfrak{s},\mathfrak{f},\lambda\}} f^{\{\mathfrak{f}\}}\left(Y_j^{\{\mathfrak{f},\lambda\}}\right), \\
Y_i^{\{\mathfrak{f},\lambda\}} &= y_n + h \sum_{l=1}^{\lambda-1} \sum_{j=1}^{s^{\{\mathfrak{f}\}}} b_j^{\{\mathfrak{f}\}} f^{\{\mathfrak{f}\}}\left(Y_j^{\{\mathfrak{f},l\}}\right) + \\
&\quad + H \sum_{j=1}^{s^{\{\mathfrak{s}\}}} a_{i,j}^{\{\mathfrak{f},\mathfrak{s},\lambda\}} f^{\{\mathfrak{s}\}}\left(Y_j^{\{\mathfrak{s}\}}\right) + \\
&\quad + h \sum_{j=1}^{s^{\{\mathfrak{f}\}}} a_{i,j}^{\{\mathfrak{f},\mathfrak{f}\}} f^{\{\mathfrak{f}\}}\left(Y_j^{\{\mathfrak{f},\lambda\}}\right), \\
&\quad \lambda = 1, \dots, M, \\
y_{n+1} &= y_n + h \sum_{\lambda=1}^{M} \sum_{i=1}^{s^{\{\mathfrak{f}\}}} b_i^{\{\mathfrak{f}\}} f^{\{\mathfrak{f}\}}\left(Y_i^{\{\mathfrak{f},\lambda\}}\right) + \\
&\quad + H \sum_{j=1}^{s^{\{\mathfrak{s}\}}} b_i^{\{\mathfrak{s}\}} f^{\{\mathfrak{s}\}}\left(Y_i^{\{\mathfrak{s}\}}\right).
\end{aligned}$$

The base schemes are Runge-Kutta methods, $(A^{\{\mathfrak{f},\mathfrak{f}\}}, b^{\{\mathfrak{f}\}})$ for the slow component and $(A^{\{\mathfrak{s},\mathfrak{s}\}}, b^{\{\mathfrak{s}\}})$ for the fast component. The coefficients $A^{\{\mathfrak{s},\mathfrak{f},\lambda\}}$, $A^{\{\mathfrak{f},\mathfrak{s},\lambda\}}$ realize the coupling between the two components.

1.1 Order conditions

The MGARK scheme can be written as a GARK scheme [3] over the macro-step H with the fast stage vectors $Y^{\{\mathfrak{f}\}} := [Y^{\{\mathfrak{f},1\}\,T}, \dots, Y^{\{\mathfrak{f},M\}\,T}]^T$. The corresponding Butcher tableau reads

$$\begin{array}{cccc|c}
\frac{1}{M}A^{\{\mathfrak{f},\mathfrak{f}\}} & 0 & \cdots & 0 & A^{\{\mathfrak{f},\mathfrak{s},1\}} \\
\frac{1}{M}\mathbf{1}b^{\{\mathfrak{f}\}\,T} & \frac{1}{M}A^{\{\mathfrak{f},\mathfrak{f}\}} & \cdots & 0 & A^{\{\mathfrak{f},\mathfrak{s},2\}} \\
\vdots & & \ddots & & \vdots \\
\frac{1}{M}\mathbf{1}b^{\{\mathfrak{f}\}\,T} & \frac{1}{M}\mathbf{1}b^{\{\mathfrak{f}\}\,T} & \cdots & \frac{1}{M}A^{\{\mathfrak{f},\mathfrak{f}\}} & A^{\{\mathfrak{f},\mathfrak{s},M\}} \\
\hline
\frac{1}{M}A^{\{\mathfrak{s},\mathfrak{f},1\}} & \frac{1}{M}A^{\{\mathfrak{s},\mathfrak{f},2\}} & \cdots & \frac{1}{M}A^{\{\mathfrak{s},\mathfrak{f},M\}} & A^{\{\mathfrak{s},\mathfrak{s}\}} \\
\hline
\frac{1}{M}b^{\{\mathfrak{f}\}\,T} & \frac{1}{M}b^{\{\mathfrak{f}\}\,T} & \cdots & \frac{1}{M}b^{\{\mathfrak{f}\}\,T} & b^{\{\mathfrak{s}\}\,T}
\end{array}$$

Therefore the order conditions for MGARK schemes can be derived from the corresponding ones for GARK schemes. Up to order two the order conditions given in Table 1 have to be fulfilled.

1.2 Stability

We consider systems (1) where each of the component functions is dispersive (with constants $\nu^{\{\mathfrak{s}\}} < 0$, $\nu^{\{\mathfrak{f}\}} < 0$):

p	order condition
1	$b^{\{\mathfrak{s}\}\,T}\mathbb{1} = 1$
	$b^{\{\mathfrak{f}\}\,T}\mathbb{1} = 1$
2	$b^{\{\mathfrak{s}\}\,T}A^{\{\mathfrak{s},\mathfrak{s}\}}\mathbb{1} = \frac{1}{2}$
	$b^{\{\mathfrak{s}\}\,T}\left(\sum_{\lambda=1}^{M}A^{\{\mathfrak{s},\mathfrak{f},\lambda\}}\right)\mathbb{1} = \frac{M}{2}$
	$b^{\{\mathfrak{f}\}\,T}A^{\{\mathfrak{f},\mathfrak{f}\}}\mathbb{1} = \frac{1}{2}$
	$b^{\{\mathfrak{f}\}\,T}\left(\sum_{\lambda=1}^{M}A^{\{\mathfrak{f},\mathfrak{s},\lambda\}}\mathbb{1}\right) = \frac{M}{2}$

Table 1. Order conditions for MGARK schemes.

$$\left\langle f^{\{\mathfrak{s}\}}(y) - f^{\{\mathfrak{s}\}}(z)\,, y-z\right\rangle \leq \nu^{\{\mathfrak{s}\}}\,\|y-z\|^2,$$
$$\left\langle f^{\{\mathfrak{f}\}}(y) - f^{\{\mathfrak{f}\}}(z)\,, y-z\right\rangle \leq \nu^{\{\mathfrak{f}\}}\,\|y-z\|^2,$$

with respect to the same scalar product $\langle\cdot,\cdot\rangle$. As for two solutions $y(t)$ and $\widetilde{y}(t)$ of (1), each starting from a different initial condition, the norm of the solution difference $\Delta y(t) = \widetilde{y}(t) - y(t)$ is non-increasing, we demand a similar property from the numerical approximationos: the MGARK scheme is said to be nonlinearly stable, if the inequality

$$\|y_{n+1} - \tilde{y}_{n+1}\| \leq \|y_n - \tilde{y}_n\|$$

holds for any two numerical approximations y_{n+1} and $\widetilde{y}_{n+1}$ obtained by applying the scheme to the ODE (1) with dispersive functions and with initial values y_n and $\widetilde{y}_n$.

As a consequence of stablity theory for GARK schemes, an MGARK scheme applied to a component-wise partitioned right-hand side is nonlinearily stable, if both base schemes are algebraically stable.

1.3 Two simple MGARK schemes for multipysics application

In general, one is interested in a rough approximation of coupled multiphysics problems, which reflect the impact of the couplings of both systems. Hence we restrict to MGARK schemes of order 2:

- MGARK-IMEX-2: The implicit-explicit version solves the fast, stiff part with an implicit base scheme, and the slow, non-stiff part with an explicit one. The coefficients are given by

$$b^{\{\mathfrak{s}\}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad A^{\{\mathfrak{s},\mathfrak{s}\}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$A^{\{\mathfrak{s},\mathfrak{f},1\}} = \begin{pmatrix} 0 \\ M \end{pmatrix},$$
$$A^{\{\mathfrak{s},\mathfrak{f},\lambda\}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \forall \lambda = 2,\ldots,M,$$
$$b^{\{\mathfrak{f}\}} = 1, \quad A^{\{\mathfrak{f},\mathfrak{f}\}} = \frac{1}{2},$$
$$A^{\{\mathfrak{f},\mathfrak{s},\lambda\}} = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \quad \forall \lambda = 1,\ldots,M.$$

  Note that only the fast part is algebraically stable, but neither the slow part and the joint system.
- MGARK-IMIM-2: To get an overall stable scheme, both parts are solved by an implicit base scheme. The coefficients are given by

$$b^{\{\mathfrak{s}\}} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad A^{\{\mathfrak{s},\mathfrak{s}\}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$
$$A^{\{\mathfrak{s},\mathfrak{f},1\}} = \begin{pmatrix} 0 \\ \frac{M}{2} \end{pmatrix},$$
$$A^{\{\mathfrak{s},\mathfrak{f},\lambda\}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \forall \lambda = 2,\ldots,M,$$
$$b^{\{\mathfrak{f}\}} = 1, \quad A^{\{\mathfrak{f},\mathfrak{f}\}} = \frac{1}{2},$$
$$A^{\{\mathfrak{f},\mathfrak{s},\lambda\}} = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \quad \forall \lambda = 1,\ldots,M.$$

As both base schemes are algebraically stable, the MGARK method inherits this property for a component-wise partitioning.

2 Benchmark example

We will test both MGARK implementations for the electrical-thermal multiphysics system introduced in [1] with the specifications discussed in [2]. The thermal component defines the slow (and non-stiff) part, the electrical component the fast (and stiff) part of the system.

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