Jacobian Structure of Coupled Electromagnetic Field and Lumped Circuit Models

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Summary. Motivated by the aim of an efficient coupled electromagnetic field and lumped circuit simulation, we show that one can form the model equations in such a way that the discretized equation system (using FIT method for spatial and BDF method for time discretization) has an exploitable Jacobian structure.

1 Electromagnetic Field Model

The electromagnetic fields can be described by the full-wave Maxwell's equations

$$\nabla\cdot\mathbf{D}=\rho,\qquad \nabla\times\mathbf{E}=-\partial_t\mathbf{B}$$
$$\nabla\cdot\mathbf{B}=0,\qquad \nabla\times\mathbf{H}=\mathbf{J}+\partial_t\mathbf{D}$$

equipped with the material laws

$$\mathbf{D}=\varepsilon\mathbf{E},\qquad \mathbf{H}=\nu\mathbf{B},$$

where **D**, **E**, **B**, **H**, **J** and ρ are the displacement field, electric field, magnetic induction, magnetic field, free current density and charge density. The material dependent parameters ε and μ are the permittivity and the magnetic permeability. The charge ρ and the current density **J** can be described by the following model equations:

$$\rho=\begin{cases}0 & \text{for metal and isolator}\\ q(n-p-N_D) & \text{for semiconductor}\end{cases}\tag{1}$$

and

$$\mathbf{J}=\begin{cases}\sigma\mathbf{E} & \text{for metal}\\ \mathbf{J_n}+\mathbf{J_p} & \text{for semiconductor}\\ 0 & \text{for isolator}\end{cases}\tag{2}$$

with the electron and hole current densities J_n and J_p as well as the electron and hole concentrations n and p satisfying

$$\partial_t n-\nabla\cdot\mathbf{J_n}+qR(n,p)=0\tag{3}$$
$$\partial_t p+\nabla\cdot\mathbf{J_p}+qR(n,p)=0\tag{4}$$

with

$$\mathbf{J_n}=qD_n\nabla n-q\mu_n n\mathbf{E},\quad \mathbf{J_p}=qD_p\nabla p+q\mu_p p\mathbf{E}.$$

The material depending parameters N_D, σ, μ_n and μ_p describe the doping concentration, the conductivity, the mobility of electrons and the mobility of holes. The function R gives the recombination rate for electrons and holes. Finally, q is the elementary charge and D_n, D_p are the diffusion coefficents.

Notice, the semiconductor current density model reflects the drift-diffusion model [9] and should be extended by an additional current density part caused by the self-induced Lorentz force in case of circuits with fast-transient signals, see [8].

To facilitate the coupling between the electromagnetic field simulation with a lumped circuit simulation, the Maxwell equations are written in potential form using the scalar potential φ and the vector potential **A** [1,2] satisfying

$$\mathbf{B}=\nabla\times\mathbf{A},\qquad \nabla\varphi=-\mathbf{E}-\partial_t\mathbf{A}.\tag{5}$$

The existence of these potentials follows from the Gauß' law $\nabla\cdot\mathbf{B}=0$ for magnetism and the Maxwell-Faraday law $\nabla\times\mathbf{E}=-\partial_t\mathbf{B}$. For uniqueness of **A** and φ, we need a gauge condition. Because of numerical stability reasons [3], we choose the Lorenz gauging

$$\nabla\cdot\mathbf{A}+c\partial_t\varphi=0\tag{6}$$

with a suitable constant c. Using (5), the full Maxwell equations reduce to

$$\nabla\cdot(\varepsilon\nabla\varphi+\varepsilon\partial_t\mathbf{A})=-\rho\tag{7}$$
$$\nabla\times(\nu\nabla\times\mathbf{A})+\partial_t(\varepsilon\nabla\varphi+\varepsilon\partial_t\mathbf{A})=\mathbf{J}\tag{8}$$

with ρ and **J** given by (1) and (2) in which **E** is replaced by $-\nabla\varphi-\partial_t\mathbf{A}$. Finally, a new variable, the pseudo-canonical momentum $\Pi=\partial_t\mathbf{A}$ is introduced to avoid the second-order time derivative [7].

2 Lumped Circuit Equations

For lumped circuit models, the Kirchhoff's laws are satisfied and can be written as

$$A\mathbf{i}=0,\qquad \mathbf{v}=A^\top\mathbf{e}\tag{9}$$

with the incidence matrix A mapping branches to nodes of the circuit. The circuit variables are the vector **i** of all branch currents, the vector **v** of all branch voltages and the vector $\mathbf{e}$ of all nodal potentials. In contrast to the field variables, the circuit variables depend on time t only. Additionally, we have the constitutive element equations

$$\mathbf{i}_1 = \tfrac{\mathrm{d}}{\mathrm{d}t} q(\mathbf{v}_1,t) + g(\mathbf{v}_1,t), \quad \mathbf{v}_2 = \tfrac{\mathrm{d}}{\mathrm{d}t} \phi(\mathbf{i}_2,t) + r(\mathbf{i}_2,t)$$

for lumped current and voltage controlling elements, respectively. Notice, all basic types as capacitances, inductances, resistances and sources are covered by a suitable choice of the functions q, g, ϕ and r.

Splitting the branches of the incidence matrix into $A = (A_1, A_2, A_3)$ with respect to the current controlling, voltage controlling and electromagnetic field element models, the circuit equations can be written in the compact form of the Modified Nodal Analysis (MNA) as [6, 7]

$$A_1 \tfrac{\mathrm{d}}{\mathrm{d}t} q(A_1^\top \mathbf{e},t) + A_1 g(A_1^\top \mathbf{e},t) + A_2 \mathbf{i}_2 + A_3 \mathbf{i}_3 = 0 \quad (10)$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \phi(\mathbf{i}_2,t) + r(\mathbf{i}_2,t) - A_2^\top \mathbf{e} = 0 \quad (11)$$

together with $\mathbf{v}_3 = A_3^\top \mathbf{e}$.

3 Interface Model

We assume the interface between the electromagnetic field model and the lumped circuit model to be perfectly electric conducting such that $\mathbf{B} \cdot n_\perp = 0$ and $\mathbf{E} \cdot n_\parallel = 0$ with $n_\perp$ and $n_\parallel$ being the outer unit normal vectors transversal and parallel to the contact boundary. This motivates the boundary conditions [3]

$$(\nabla \times \mathbf{A}) \cdot n_\perp = 0, \qquad (\nabla \varphi) \cdot n_\parallel = 0. \qquad (12)$$

Denoting by Γ_k the k-th contact of the electromagnetic field model element with Γ_0 being the reference contact and choosing any position $x^k \in \Gamma_k$, we obtain the coupling equations

$$\mathbf{i}_3^k = \int_{\Gamma_k} [\mathbf{J} - \partial_t(\varepsilon(\nabla\varphi + \Pi))] \cdot n_\perp \, \mathrm{d}\sigma$$

$$\mathbf{v}_3^k = \varphi(x^k) - \varphi(x^0)$$

that can be bundled as

$$\mathbf{i}_3 = B_\mathbf{J} \mathbf{J} + B_\varphi \partial_t \varphi + B_\Pi \partial_t \Pi, \qquad (13)$$

$$A_3^\top \mathbf{e} = R_\varphi \varphi. \qquad (14)$$

with linear boundary operators $B_\mathbf{J}$, B_φ, B_Π and R_φ.

4 Coupled Model Structure

Discretizing the electromagnetic field model in space by the FIT discretization as described in [2, 3] and using as time discretization the BDF methods for the resulting differential algebraic system as given in [5], we obtain a Jacobian structure of the form

$$J = \begin{pmatrix} J_E & J_{EB} & 0 \\ J_{BE} & I & J_{BC} \\ 0 & J_{CB} & J_C \end{pmatrix}$$

with a diagonally dominant matrix J_E for the electromagnetic and a positiv semidefinite matrix J_C for the lumped circuit part, respectively, if the time steps and mesh size are sufficiently small and if we take the variable order φ, $\mathbf{A}$, $\mathbf{J}$, n, p, $\mathbf{i}_3$, $\mathbf{e}$, $\mathbf{i}_2$ as well as the coupled equation system order (7), (8), (2), (3), (4), (13), (10), (11). Some details about J_C and J_E are given in [3, 4]. Important is here that we plug in the discretized versions of the equations (12), (14), (1), (6) and $\Pi = \partial_t A$ before. It allows to combine an efficient iterative solver for the high dimensional (due to 3D discretization) matrix part J_E resolving φ, $\mathbf{A}$, $\mathbf{J}$, n, p with a simple evaluation process for the determination of the coupling current i_3 and a direct solver for the elimination of the circuit variables $\mathbf{e}$ and $\mathbf{i}_2$ after use of a Schur complement approach.

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